

Approximate Solutions, Existence, and Uniqueness of the Cauchy Problem of Fuzzy Differential Equations*

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The Cauchy problem of fuzzy differential equations is investigated by the use of the concept of H -differentiability due to M. L. Puri and D. A. Ralescu (*J. Math. Anal. Appl.* **91**, 1983, 552–558). The existence and uniqueness theorem is obtained for the solution of the Cauchy problem $x'(t) = f(t, x(t))$, $x(t_0) = x_0$, for the fuzzy-valued mappings of a real variable whose values are normal, convex, upper semicontinuous, and compactly supported fuzzy sets in R^n , where the function f satisfies the generalized Lipschitz condition. © 1996 Academic Press, Inc.

1. INTRODUCTION

Kaleva [8] discussed the properties of differentiable fuzzy valued mappings by means of the concept of H -differentiability due to Puri and Ralescu [13]. Seikkala [16] defined the fuzzy derivative which is a generalization of the Hukuhara derivative in [13] and the fuzzy integral which is the same as that proposed by Dubois and Prade [5, 6]. By the use of the extension principle, Seikkala also showed that the fuzzy initial value problem $x' = f(t, x(t))$, $x(t_0) = 4x_0$ has a unique fuzzy solution when f satisfies the generalized Lipschitz condition which guarantees a unique

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solution to the deterministic initial value problem. Kaleva [9] also studied the Cauchy problem of fuzzy differential equations.

In this paper, by the use of the concept of H -differentiability, we study the Cauchy problem of fuzzy differential equations for the fuzzy valued mappings of a real variable whose values are normal, convex, upper semicontinuous, and compactly supported fuzzy sets in R^n . Several important results are obtained by applying the embedding theorem in [9] which is a generalization of the classical Radström embedding results [3, 15].

As preliminaries we recall some basic results on fuzzy number and the measurability, integrability, and differentiability properties for the fuzzy set-valued mappings in [8]. We list several comparison theorems on classical ordinary differential equations in [11]. In Section 3, we show the relation between a solution and its approximate solution to the Cauchy problem of the fuzzy differential equation, and furthermore, in Section 4, we prove the existence and uniqueness theorem for a solution to the Cauchy problem of the fuzzy differential equation $x'(t) = f(t, x(t))$, $x(t_0) = x_0$ when f satisfies the generalized Lipschitz condition.

2. PRELIMINARIES

Let $P_k(R^n)$ denote the family of all nonempty compact convex subset of R^n and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Let A and B be two nonempty bounded subsets of R^n . The distance between A and B is defined by the Hausdorff metric,

$$d(A, B) = \max[\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|],$$

where $\|\cdot\|$ denotes the usual Euclidean norm in R^n . Then it is clear that $(P_k(R^n), d)$ becomes a metric space. The following theorem gives a more precise result (see [4]).

PROPOSITION 2.1. *The metric space $(P_k(R^n), d)$ is complete and separable.*

Let $T = [t_0, t_0 + p] \subset R$ ($p > 0$) be a compact interval and denote

$$E^n = \{u: R^n \rightarrow [0, 1] \mid u \text{ satisfies (i)–(iv) below}\},$$

where

- (i) u is normal, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,
- (ii) u is fuzzy convex,
- (iii) u is upper semicontinuous,
- (iv) $[u]^0 = \text{cl}\{x \in R^n \mid u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$ denote $[u]^\alpha = \{x \in R^n \mid u(x) \geq \alpha\}$. Then from (i)–(iv), it follows that the α -level set $[u]^\alpha \in P_k(R^n)$ for all $0 \leq \alpha \leq 1$.

If $g: R^n \times R^n \rightarrow R^n$ is a function, then according to Zadeh's extension principle we can extend g to $E^n \times E^n \rightarrow E^n$ by the equation

$$g(u, v)(z) = \sup_{z=g(x, y)} \min\{u(x), v(y)\}.$$

It is well known that

$$[g(u, v)]^\alpha = g([u]^\alpha, [v]^\alpha)$$

for all $u, v \in E^n$, $0 \leq \alpha \leq 1$, and continuous function g . Especially for addition and scalar multiplication, we have

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha,$$

where $u, v \in E^n$, $k \in R$, $0 \leq \alpha \leq 1$.

Furthermore, we have the following representation theorem (see [12]).

PROPOSITION 2.2. *If $u \in E^n$, then*

- (1) $[u]^\alpha \in P_k(R^n)$ for all $0 \leq \alpha \leq 1$,
- (2) $[u]^{\alpha_2} \subset [u]^{\alpha_1}$ for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$,
- (3) If $\{\alpha_k\} \subset [0, 1]$ is a nondecreasing sequence converging to $\alpha > 0$, then

$$[u]^\alpha = \bigcap_{k \geq 1} [u]^{\alpha_k}.$$

Conversely, if $\{A^\alpha \mid 0 \leq \alpha \leq 1\}$ is a family of subsets of R^n satisfying (1)–(3), then

$$[u]^\alpha = A^\alpha \quad \text{for } 0 < \alpha \leq 1$$

and

$$[u]^0 = \overline{\bigcup_{0 < \alpha \leq 1} A^\alpha} \subset A^0.$$

Define $D: E^n \times E^n \rightarrow R^+ \cup \{0\}$ by the equation

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha),$$

where d is the Hausdorff metric defined in $P_K(R^n)$. Then it is easy to show that D is a metric in E^n .

Using the results in [14], we know that

- (1) (E^n, D) is a complete metric space;
- (2) $D(u + w, v + w) = D(u, v)$ for all $u, v, w \in E^n$;
- (3) $D(ku, kv) = |k|D(u, v)$ for all $u, v \in E^n, k \in R$.

We recall some measurability, integrability, and differentiability properties for the fuzzy set-valued mappings in [8].

DEFINITION 2.3. We say that a mapping $F: T \rightarrow E^n$ is strongly measurable if for all $a \in [0, 1]$ the set-valued mapping $F_a: T \rightarrow P_k(R^n)$ defined by

$$F_a(t) = [F(t)]^\alpha$$

is (Lebesgue) measurable, when $P_k(R^n)$ is endowed with the topology generated by the Hausdorff metric d .

A mapping $F: T \rightarrow E^n$ is called integrably bounded if there exists an integrable function h such that $\|x\| \leq h(t)$ for all $x \in F_0(t)$.

DEFINITION 2.4. Let $F: T \rightarrow E^n$. The integral of F over T , denoted by $\int_T F(t) dt$, is defined level-wise by the equation

$$\begin{aligned} \left[\int_T F(t) dt \right]^\alpha &= \int_T F_a(t) dt \\ &= \left\{ \int_T f(t) dt \mid f: T \rightarrow R^n \text{ is a measurable selection for } F_a \right\} \end{aligned}$$

for all $0 < \alpha \leq 1$.

A strongly measurable and integrably bounded mapping $F: T \rightarrow E^n$ is said to be integrable over T if $\int_T F(t) dt \in E^n$.

PROPOSITION 2.5. If $F: T \rightarrow E^n$ is strongly measurable and integrably bounded, then F is integrable.

Remark. From Remark 4.1 in [8], it is known that $[\int_T F(t) dt]^0 = \int_T F_0(t) dt$.

COROLLARY 2.6. If $F: T \rightarrow E^n$ is continuous, then it is integrable.

PROPOSITION 2.7. Let $F: T \rightarrow E^n$ be integrable and $c \in T$. Then

$$\int_a^b F(t) dt = \int_a^c F(t) dt + \int_c^b F(t) dt.$$

PROPOSITION 2.8. *Let $F, G: T \rightarrow E^n$ be integrable and $\lambda \in R$. Then*

- (I) $\int_T (F(t) + G(t)) dt = \int_T F(t) dt + \int_T G(t) dt,$
- (II) $\int_T \lambda F(t) dt = \lambda \int_T F(t) dt,$
- (III) $D(F, G)$ is integrable,
- (IV) $D(\int_T F(t) dt, \int_T G(t) dt) \leq \int_T D(F, G)(t) dt.$

Remark. Suppose $A \in E^n$ and define $F: [a, b] \rightarrow E^n$ by $F(s) = A$ for all $a \leq s \leq b$. Then from Example 4.1 in [8], we have

$$\int_a^b F(t) dt = (b - a)A.$$

Let $x, y \in E^n$. If there exists a $z \in E^n$ such that $x = y + z$, then we call z the H -difference of x and y , denoted by $x - y$.

DEFINITION 2.9. A mapping $F: T \rightarrow E^n$ is differentiable at $t_0 \in T$, if there exists a $F'(t_0) \in E^n$ such that the limits

$$\lim_{h \rightarrow 0^+} \frac{F(t_0 + h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{F(t_0) - F(t_0 - h)}{h}$$

exist and are equal to $F'(t_0)$.

Here the limit is taken in the metric space (E^n, D) . At the end point of T , we consider only the one-sided derivatives.

If $F: T \rightarrow E^n$ is differentiable at $t_0 \in T$, then we say that $F'(t_0)$ is the fuzzy derivative of $F(t)$ at the point t_0 .

PROPOSITION 2.10. *If $F: T \rightarrow E^n$ is differentiable, then it is continuous.*

Remark. If $F: T \rightarrow E^n$ is differentiable and $F'(t)$ is continuous, then we denote $F \in C^1[T, E^n]$.

PROPOSITION 2.11. *If $F, G: T \rightarrow E^n$ are differentiable and $\lambda \in R$, then $(F + G)'(t) = F'(t) + G'(t)$, $(\lambda F)'(t) = \lambda F'(t)$.*

PROPOSITION 2.12. *Let $F: T \rightarrow E^n$ be continuous. Then for every $t \in T$, the integral $G(t) = \int_{t_0}^t F(s) ds$ is differentiable and $G'(t) = F(t)$.*

PROPOSITION 2.13. *Let $F: T \rightarrow E^n$ be differentiable and assume that the derivative $F'(t)$ is integrable over T . Then for each $s \in T$ we have*

$$F(s) = F(t_0) + \int_{t_0}^s F'(t) dt.$$

PROPOSITION 2.14. *There exists a real Banach space X such that E^n can be the embedding as a convex cone C with vertex $\mathbf{0}$ into X . Furthermore the following conditions hold true:*

- (i) *the embedding j is isometric,*
- (ii) *addition in X induces addition in E^n ,*
- (iii) *multiplication by nonnegative real number in X induces the corresponding operation in E^n ,*
- (iv) *$C - C$ is dense in X ,*
- (v) *C is closed.*

Remark. If $x(t): T \rightarrow E^n$ is differentiable at $t_0 \in T$, then it is easy to see that

$$(jx)(t) = j(x(t)): T \rightarrow X$$

is Fréchet differentiable at t_0 and $(jx)'(t_0) = j(x'(t_0))$, where j is the embedding in Proposition 2.14. In the following we list several comparison theorems on classical ordinary differential equations as follows (see [11]):

PROPOSITION 2.15. *Let $G \subset \mathbb{R}^2$ be an open set and $g \in C[G, \mathbb{R}^1]$, $(t_0, u_0) \in G$. Suppose $r(t)$ is the maximum solution to the initial value problem*

$$u' = g(t, u), \quad u(t_0) = u_0, \quad (*)$$

and its largest interval of existence of right solution is $[t_0, t_0 + a)$. If $[t_0, t_1] \subset [t_0, t_0 + a)$, then there exists an $\varepsilon_0 > 0$ such that the maximum solution $r(t, \varepsilon)$ to the initial value problem

$$u' = g(t, u) + \varepsilon, \quad u(t_0) = u_0 + \varepsilon$$

exists on $[t_0, t_1]$ whenever $0 < \varepsilon < \varepsilon_0$, and $r(t, \varepsilon)$ uniformly converges to $r(t)$ on $[t_0, t_1]$ as $\varepsilon \rightarrow 0^+$.

PROPOSITION 2.16. *Let $G \subset \mathbb{R}^2$ be an open set, $g \in C[G, \mathbb{R}^1]$, $(t_0, u_0) \in G$. Suppose that the maximum solution to the initial value problem $(*)$ is $r(t)$ and its largest interval of existence of right solution is $[t_0, t_0 + a)$. If $m(t) \in C[(t_0, t_0 + a), \mathbb{R}^1]$ satisfies $(t, m(t)) \in G$ for all $t \in [t_0, t_0 + a)$, $m(t_0) \leq U_0$, and*

$$Dm(t) \leq g(t, m(t)), \quad \forall t \in [t_0, t_0 + a) \setminus \Gamma,$$

where D is one of the four Dini derivatives (see [11]), Γ at most is a countable set on $[t_0, t_0 + a)$. Then we must have

$$m(t) \leq r(t) \quad \text{for all } t \in [t_0, t_0 + a).$$

3. THE RELATION BETWEEN A SOLUTION AND ITS APPROXIMATE SOLUTIONS TO FUZZY DIFFERENTIAL EQUATIONS

Assume that $f: T \times W \rightarrow E^n$ is continuous (it is denoted by $f \in C[T \times W, E^n]$). Consider the initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad (3.0)$$

where $W \subset E^n$, $x_0 \in W$.

From Propositions 2.10, 2.12, and 2.13, it immediately follows:

LEMMA 3.1. *A mapping $x: T \rightarrow W$ is a solution to the problem (3.0) if and only if it is continuous and satisfies the integral equation*

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds$$

for all $t \in T$.

In the following we give the relation between a solution and its approximate solutions.

We denote $R_0 = [t_0, t_0 + p] \times B(x_0, q)$ where $p > 0$, $q > 0$, $x_0 \in E^n$, $B(x_0, q) = \{x \in E^n \mid D(x, x_0) \leq q\}$.

THEOREM 3.2. *Let $f \in C[R_0, E^n]$, $r \in (0, p)$, $x_n \in C^1[[t_0, t_0 + r], B(x_0, q)]$ such that*

$$jx'_n(t) = jf(t, x_n(t)) + B_n(t), \quad x_n(t_0) = x_0, \quad \|B_n(t)\| \leq \varepsilon_n, \\ \forall t \in [t_0, t_0 + r] \quad (n = 1, 2, \dots), \quad (3.1)$$

where $\varepsilon_n > 0$, $\varepsilon_n \rightarrow 0$, $B_n(t) \in C[[t_0, t_0 + r], X]$, and j is the isometric embedding from (E^n, D) onto its range in the Banach space X . For each $t \in [t_0, t_0 + r]$ there exists an $\delta(t) > 0$ such that the H -differences $x_n(t+h) - x_n(t)$ and $x_n(t) - x_n(t-h)$ exist for all $0 \leq h < \delta(t)$ and $n = 1, 2, \dots$.

If we have

$$D(x_n(t), x(t)) \rightarrow 0 \quad \text{u.c. for all } t \in [t_0, t_0 + r] (n \rightarrow \infty) \quad (3.2)$$

(u.c. denotes the uniform convergence), then $x \in C^1[[t_0, t_0 + r], B(x_0, q)]$ and

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t \in [t_0, t_0 + r]. \quad (3.3)$$

Proof. From (3.2) we know that $x(t) \in C[[t_0, t_0 + r], B(x_0, q)]$. For fixed $t_1 \in [t_0, t_0 + r]$ and any $t \in [t_0, t_0 + r]$, $t > t_1$, denote

$$F(t, n) = \frac{jx_n(t) - jx_n(t_1)}{t - t_1} - jf(t_1, x_n(t_1)) - B_n(t_1).$$

It is well known that

$$\begin{aligned} \lim_{t \rightarrow t_1 + 0} F(t, n) &= (jx_n)'(t_1) - jf(t_1, x_n(t_1)) - B_n(t_1) \\ &= jx_n'(t_1) - jf(t_1, x_n(t_1)) - B_n(t_1) = \theta(\in X) \\ \lim_{n \rightarrow \infty} F(t, n) &= \frac{jx(t) - jx(t_1)}{t - t_1} - jf(t_1, x(t_1)). \end{aligned} \quad (3.4)$$

From $f \in C^1[R_0, E^n]$ is known that for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$D(f(t, z), f(t_1, x(t_1))) < \frac{\varepsilon}{4} \quad (3.5)$$

whenever $t_1 < t < t_1 + \delta_1$ and $D(z, x(t_1)) < \delta_1$ with $z \in B(x_0, q)$.

Take natural number $N > 0$ such that

$$\varepsilon_n < \frac{\varepsilon}{4}, D(x_n(t), x(t)) < \frac{\delta_1}{2} \quad \text{for any } n > N, t \in [t_0, t_0 + r]. \quad (3.6)$$

Take $\delta > 0$ such that $\delta < \delta_1$ and

$$D(x(t), x(t_1)) < \frac{\delta_1}{2} \quad \text{whenever } t_1 < t < t_1 + \delta. \quad (3.7)$$

By the definition of $F(t, n)$ and (3.1), we have

$$jx_n(t) - jx_n(t_1) - (t - t_1)jx_n'(t_1) = (t - t_1)F(t, n). \quad (3.8)$$

We choose $\varphi \in X^*$ such that $\|\varphi\| = 1$ and

$$\begin{aligned} &\varphi(jx_n(t) - jx_n(t_1) - (t - t_1)jx_n'(t_1)) \\ &= \|jx_n(t) - jx_n(t_1) - (t - t_1)jx_n'(t_1)\|. \end{aligned}$$

Let $\psi(t) = \varphi(jx_n(t)) - (t - t_1)\varphi(jx'_n(t_1))$, consequently,

$$\psi'(t) = \varphi(jx'_n(t)) - \varphi(jx'_n(t_1))$$

hence

$$\begin{aligned} & \|jx_n(t) - jx_n(t_1) - (t - t_1)jx'_n(t_1)\| \\ &= \psi(t) - \psi(t_1) = \psi'(\bar{t})(t - t_1) \\ &= \varphi(jx'_n(\bar{t}) - jx'_n(t_1))(t - t_1) \\ &\leq \|\varphi\| \cdot \|jx'_n(\bar{t}) - jx'_n(t_1)\| \cdot (t - t_1) \\ &= \|jx'_n(\bar{t}) - jx'_n(t_1)\| \cdot (t - t_1), \end{aligned}$$

where $t_1 \leq \bar{t} \leq t$. In view of (3.8), we have

$$\|F(t, n)\| \leq \|jx'_n(\bar{t}) - jx'_n(t_1)\|, \quad t_1 \leq \bar{t} \leq t. \quad (3.9)$$

From (3.7) and (3.6) we know that

$$D(x(\bar{t}), x(t_1)) < \frac{\delta_1}{2}$$

and

$$\begin{aligned} D(x_n(\bar{t}), x(t_1)) &\leq D(x_n(\bar{t}), x(\bar{t})) + D(x(\bar{t}), x(t_1)) \\ &< \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1. \end{aligned}$$

Hence by (3.9) and (3.5) we have

$$\begin{aligned} \|F(t, n)\| &\leq \|jx'_n(\bar{t}) - jx'_n(t_1)\| \\ &= \|jf(\bar{t}, x_n(\bar{t})) + B_n(\bar{t}) - jf(t_1, x_n(t_1)) - B_n(t_1)\| \\ &\leq \|jf(\bar{t}, x_n(\bar{t})) - jf(t_1, x(t_1))\| \\ &\quad + \|jf(t_1, x(t_1)) - jf(t_1, x_n(t_1))\| + 2\varepsilon_n \\ &= D(f(\bar{t}, x_n(\bar{t})), f(t_1, x(t_1))) \\ &\quad + D(f(t_1, x(t_1)), f(t_1, x_n(t_1))) + 2\varepsilon_n \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2\varepsilon_n < \varepsilon \end{aligned}$$

whenever $n > N$ and $t_1 < t < t_1 + \delta$.

Now let $n \rightarrow \infty$, and applying Eq. (3.4), we have

$$\left\| \frac{jx(t) - jx(t_1)}{t - t_1} - jf(t_1, x(t_1)) \right\| \leq \varepsilon, \quad t_1 < t < t_1 + \delta. \quad (3.10)$$

On the other hand, from the assumption of Theorem 3.1, there exists an $\delta(t_1) \in (0, \delta)$ such that the H -differences

$$x_n(t) - x_n(t_1)$$

exist for all $t \in [t_1, t_1 + \delta(t_1)]$ and $n = 1, 2, \dots$.

Let $y_n(t) = x_n(t) - x_n(t_1)$. We verify that the fuzzy number-valued sequence $\{y_n(t)\}$ uniformly converges on $[t_1, t_1 + \delta(t_1)]$. In fact, from the assumption $D(x_n(t), x(t)) \rightarrow 0$ u.c. for all $t \in [t_0, t_0 + r]$, we know

$$\begin{aligned} D(y_n(t), y_m(t)) &= D(y_n(t) + x_n(t_1), y_m(t) + x_n(t_1)) \\ &\leq D(x_n(t), x_m(t)) + D(x_m(t), y_m(t) + x_n(t_1)) \\ &= D(x_n(t), x_m(t)) \\ &\quad + D(y_m(t) + x_m(t_1), y_m(t) + x_n(t_1)) \\ &= D(x_n(t), x_m(t)) + D(x_m(t_1), x_n(t_1)) \\ &\rightarrow 0 \quad \text{u.c. for all } t \in [t_1, t_1 + \delta(t_1)] \quad (n, m \rightarrow \infty). \end{aligned}$$

Since (E^n, D) is complete, there exists a fuzzy number-valued mapping $y(t)$ such that $\{y_n(t)\}$ uniformly converges to $y(t)$ on $[t_1, t_1 + \delta(t_1)]$ as $n \rightarrow \infty$.

In addition, we have

$$\begin{aligned} &D(x(t_1) + y(t), x(t)) \\ &\leq D(x(t_1) + y(t), x_n(t_1) + y_n(t)) + D(x_n(t_1) + y_n(t), x(t)) \\ &\leq D(x(t_1) + y(t), x(t_1) + y_n(t)) \\ &\quad + D(x(t_1) + y_n(t), x_n(t_1) + y_n(t)) + D(x_n(t), x(t)) \\ &= D(y_n(t), y(t)) + D(x_n(t_1), x(t_1)) + D(x_n(t), x(t)) \\ &\quad \forall t \in [t_1, t_1 + \delta(t_1)]. \end{aligned}$$

Let $n \rightarrow \infty$. It follows that

$$x(t_1) + y(t) \equiv x(t) \quad \text{for all } t \in [t_1, t_1 + \delta(t_1)].$$

Hence the H -differences $x(t) - x(t_1)$ exist for all $t \in [t_1, t_1 + \delta(t_1)]$.

Thus from (3.10) we have

$$D\left(\frac{x(t) - x(t_1)}{t - t_1}, f(t_1, x(t_1))\right) \leq \varepsilon, \quad t_1 < t \leq t_1 + \delta(t_1).$$

So, $\lim_{t \rightarrow t_1^+} ((x(t) - x(t_1))/(t - t_1)) = f(t_1, x(t_1))$. Similarly, we have

$$\lim_{t \rightarrow t_1^-} \frac{x(t_1) - x(t)}{t_1 - t} = f(t_1, x(t_1)).$$

Hence $x'(t_1)$ exists and

$$x'(t_1) = f(t_1, x(t_1)).$$

From $t_1 \in [t_0, t_0 + r]$ is arbitrary, we know that Eq. (3.3) holds true and

$$x \in C^1[[t_0, t_0 + r], B(x_0, q)].$$

Thus, we conclude the proof.

COROLLARY 3.3. *If we replace condition (3.1) by*

$$jx'_{n+1}(t) = jf(t, x_n(t)) + B_n(t), \quad x_n(t_0) = x_0, \quad \|B_n(t)\| \leq \varepsilon_n,$$

whenever $t \in [t_0, t_0 + r]$ ($n = 1, 2, \dots$), and retain other assumptions, then the conclusions also hold true.

Proof. This is completely similar to the proof of Theorem 3.1, hence it is omitted here.

4. EXISTENCE AND UNIQUENESS THEOREM FOR A SOLUTION

In this section we give the following conclusions.

THEOREM 4.1. *Let (a) $f \in C[R_0, E^n]$ and $D(f(t, x), \hat{0}) \leq M$, for all $(t, x) \in R_0$; (b) $g \in C[[t_0, t_0 + p] \times [0, q], R]$, $g(t, 0) \equiv 0$, and $0 \leq g(t, u) \leq M_1$, for all $t \in [t_0, t_0 + p]$, $0 \leq u \leq q$ such that $g(t, u)$ is nondecreasing on u (i.e., $t_0 \leq t \leq t_0 + p$, $0 \leq u_1 \leq u_2 \leq q \Rightarrow g(t, u_1) \leq g(t, u_2)$), the initial value problem*

$$u'(t) = g(t, u(t)), \quad u(t_0) = 0 \quad (4.1)$$

has only the solution $u(t) \equiv 0$ on $[t_0, t_0 + p]$; (c) $D(f(t, x), f(t, y)) \leq g(t, D(x, y))$, for all $(t, x), (t, y) \in R_0$, and $D(x, y) \leq q$.

Then the Cauchy problem (3.3) has unique solution $x \in C^1[[t_0, t_0 + r], B(x_0, q)]$ on $[t_0, t_0 + r]$, where $r = \min\{p, q/M, q/M_1\}$, and the successive iterations

$$x_0(t) \equiv x_0, x_{n+1}(t) = x_0 + \int_{t_0}^t f(s, x_n(s)) ds \quad (n = 0, 1, 2, \dots) \quad (4.2)$$

uniformly converge to $x(t)$ on $[t_0, t_0 + r]$.

Proof. From (4.2) and the assumption (a), by the inductive method we know

$$\begin{aligned} D(x_{n+1}(t), x_0) &\leq \int_{t_0}^t D(f(s, x_n(s)), \hat{0}) ds \\ &\leq q, \quad \forall t \in [t_0, t_0 + r], n = 0, 1, 2, \dots \end{aligned}$$

Hence $x_{n+1} \in C^1[[t_0, t_0 + r], B(x_0, q)]$ and

$$x'_{n+1}(t) = f(t, x_n(t)), \quad x_n(t_0) = x_0 \quad (n = 0, 1, 2, \dots). \quad (4.3)$$

Let $M_2 = \max\{M, M_1\}$. Then $r = \min\{p, q/M_2\}$ and we get the successive iterations as

$$\begin{cases} u_0(t) = M_2(t - t_0), & t_0 \leq t \leq t_0 + r, \\ u_{n+1}(t) = \int_{t_0}^t g(s, u_n(s)) ds, & t_0 \leq t \leq t_0 + r \quad (n = 0, 1, 2, \dots). \end{cases} \quad (4.4)$$

It is immediate that

$$u_1(t) = \int_{t_0}^t g(s, u_0(s)) ds \leq M_1(t - t_0) \leq u_0(t) \leq q, \quad \forall t \in [t_0, t_0 + r].$$

So, by the inductive method and in view that $g(t, u)$ is nondecreasing on u , we have

$$0 \leq u_{n+1}(t) \leq u_n(t) \leq q, \quad \forall t \in [t_0, t_0 + r] \quad (n = 0, 1, 2, \dots). \quad (4.5)$$

As $|u'_{n+1}(t)| = |g(t, u_n(t))| \leq M_1$, from the Ascoli-Arzelà theorem and (4.5) we know that $\{u_n(t)\}$ uniformly converges to some continuous func-

tion $u(t)$ on $[t_0, t_0 + r]$ and

$$u(t) = \int_{t_0}^t g(s, u(s)) ds. \quad (4.6)$$

Thus $u \in C^1[[t_0, t_0 + r], [0, q]]$ (in the usual sense) and u is the solution of the initial value problem (4.1). From assumption (b) we get $u(t) \equiv 0$. In addition, we have

$$\begin{aligned} D(x_1(t), x_0(t)) &= D\left(\int_{t_0}^t f(s, x_0) ds, \hat{0}\right) \\ &\leq \int_{t_0}^t D(f(s, x_0), \hat{0}) ds \\ &\leq M(t - t_0) \leq u_0(t). \end{aligned}$$

Suppose $D(x_k(t), x_{k-1}(t)) \leq u_{k-1}(t)$, then by the assumption (c), we have

$$\begin{aligned} D(x_{k+1}(t), x_k(t)) &= D\left(\int_{t_0}^t f(s, x_k(s)) ds, \int_{t_0}^t f(s, x_{k-1}(s)) ds\right) \\ &\leq \int_{t_0}^t D(f(s, x_k(s)), f(s, x_{k-1}(s))) ds \\ &\leq \int_{t_0}^t g(s, D(x_k(s), x_{k-1}(s))) ds \\ &\leq \int_{t_0}^t g(s, u_{k-1}(s)) ds = u_k(t). \end{aligned}$$

Thus by the inductive method we know

$$D(x_{n+1}(t), x_n(t)) \leq u_n(t), \quad \forall t \in [t_0, t_0 + r] \quad (n = 0, 1, 2, \dots). \quad (4.7)$$

So, we have

$$\begin{aligned} D(x'_{n+1}(t), x'_n(t)) &= D(f(t, x_n(t)), f(t, x_{n-1}(t))) \\ &\leq g(t, D(x_n(t), x_{n-1}(t))) \\ &\leq g(t, u_{n-1}(t)). \end{aligned} \quad (4.8)$$

Assume $m \geq n$, and in view of (4.8) and (4.5) we get

$$\begin{aligned}
 & D(x'_n(t), x'_m(t)) \\
 & \leq D(f(t, x_{n-1}(t)), f(t, x_n(t))) + D(f(t, x_n(t)), f(t, x_m(t))) \\
 & \quad + D(f(t, x_m(t)), f(t, x_{m-1}(t))) \\
 & \leq g(t, u_{n-1}(t)) + g(t, D(x_n(t), x_m(t))) + g(t, x_{m-1}(t)) \\
 & \leq g(t, D(x_n(t), x_m(t))) + 2g(t, u_{n-1}(t)).
 \end{aligned}$$

Furthermore, from

$$\begin{aligned}
 & D(x_n(t+h), x_m(t+h)) \\
 & \leq D(x_n(t+h), x_m(t+h) - x_m(t) + x_n(t)) \\
 & \quad + D(x_m(t+h) - x_m(t) + x_n(t), x_m(t+h)) \\
 & = D(x_n(t+h) - x_n(t), x_m(t+h) - x_m(t)) + D(x_n(t), x_m(t)),
 \end{aligned}$$

we deduce that

$$\begin{aligned}
 D^+ D(x_n(t), x_m(t)) &= \overline{\lim}_{h \rightarrow 0^+} \frac{D(x_n(t+h), x_m(t+h)) - D(x_n(t), x_m(t))}{h} \\
 &\leq \overline{\lim}_{h \rightarrow 0^+} \frac{D(x_n(t+h) - x_n(t), x_m(t+h) - x_m(t))}{h} \\
 &= D(x'_n(t), x'_m(t)) \\
 &< g(t, D(x_n(t), x_m(t))) + 2g(t, u_{n-1}(t)).
 \end{aligned}$$

Since $g(t, u_{n-1}(t))$ uniformly converges to 0, then for arbitrary $\varepsilon > 0$, there exists a natural number N such that

$$D^+ D(x_n(t), x_m(t)) < g(t, D(x_n(t), (x_m(t)))) + \varepsilon, \quad \forall m \geq n > N.$$

Here D^+ is the Dini derivative (see [11]). From the fact that $D(x_n(t_0), x_m(t_0)) = 0 < \varepsilon$ and by Theorem 2.16, we have

$$D(x_n(t), x_m(t)) \leq \omega(t, \varepsilon), \quad \forall t \in [t_0, t_0 + r], m \geq n > N, \quad (4.9)$$

where $\omega(t, \varepsilon)$ is the maximum solution to the initial value problem

$$u'(t) = g(t, u(t)) + \varepsilon, \quad u(t_0) = \varepsilon. \quad (4.10)$$

By Theorem 2.15 we know that $\omega(t, \varepsilon)$ uniformly converges to the maximum solution $u(t) \equiv 0$ of problem (4.1) on $t_0 \leq t \leq t_0 + r$ as $\varepsilon \rightarrow 0$.

Thus, according to (4.9) and that (E^n, D) is complete, we know that there exists a fuzzy set-valued mapping $x: T \rightarrow E^n$ such that $D(x_n(t), x(t))$ uniformly converges to 0 as $n \rightarrow \infty$. Applying (4.3) and Corollary 3.1 we have $x \in C^1[[t_0, t_0 + r], B(x_0, q)]$ and $x(t)$ is the solution of the initial value problem (3.3).

Finally, we prove the uniqueness. Suppose $y(t)$ is another solution of initial value problem (3.3). Let

$$m(t) = D(x(t), y(t)).$$

Then $m(t_0) = 0$ and

$$\begin{aligned} D^+m(t) &\leq D(x'(t), y'(t)) \\ &= D(f(t, x(t)), f(t, y(t))) \\ &\leq g(t, m(t)). \end{aligned}$$

Hence from Theorem 2.16 we know

$$D(x(t), y(t)) \leq u(t) \equiv 0, \quad \forall t \in [t_0, t_0 + r],$$

where $u(t) \equiv 0$ is the maximum solution of problem (4.1) on $[t_0, t_0 + r]$. Therefore $x(t) \equiv y(t)$.

COROLLARY 4.2. *Let $f \in C[R_0, E^n]$ such that $D(f(t, x), \hat{0}) \leq M$, $\forall (t, x) \in R_0$ and f satisfies the Lipschitz condition*

$$D(f(t, x), f(t, y)) \leq L \cdot D(x, y), \quad \forall (t, x), (t, y) \in R_0,$$

where $L > 0$ is a constant. Then the Cauchy problem (3.3) has unique solution $x \in C^1[[t_0, t_0 + r], B(x_0, q)]$, where $r = \min\{p, q/M, 1/L\}$ and the successive iterations (4.2) uniformly converge to $x(t)$ on $[t_0, t_0 + r]$.

Proof. In the proof of Theorem 4.1, taking $g(t, u) = L \cdot u$, we then obtain the proof of Corollary 4.1, where $m_1 = L \cdot q$, hence $r = \min\{p, q/M, 1/L\}$.

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